Dual solutions of the Greenspan–Carrier equations

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This paper concerns the boundary layer on a semi-infinite flat plate in a uniform stream of conducting fluid, with magnetic field in the stream direction. It is found that when β, the square of the ratio of the Alfvén speed to the undisturbed fluid speed, is slightly less than unity, two solutions of the Greenspan–Carrier equations governing the motion exist for $\epsilon(=\sigma\mu\nu)$ less than unity.

1. Introduction

The steady two-dimensional flow of a viscous, incompressible, electrically-conducting fluid near a semi-infinite, rigid flat plate has been shown by Greenspan & Carrier (1959) to be governed by the boundary-layer equations

$$f'''+ff''-\beta gg''=0, \tag{1.1}$$

$$g''+\epsilon(fg'-f'g)=0, \tag{1.2}$$

with boundary conditions

$$f(0)=f'(0)=g(0)=0, \quad f'(\infty)=g'(\infty)=2. \tag{1.3}$$

Here primes denote differentiation with respect to the Blasius non-dimensional variable $\eta=\frac{1}{2}(U_0/\nu x)^{\frac{1}{2}}y$, y being measured normal to the plate and x along the plate perpendicular to the leading edge. The undisturbed fluid velocity U_0 is parallel to the plate and in the x-direction and the applied magnetic field H_0 is uniform and in the same direction. The square of the ratio of the Alfvén speed to the fluid speed in the undisturbed flow is $\beta=\mu H_0^2/\rho U_0^2$, and $\epsilon=\sigma\mu\nu$, where σ is the electrical conductivity, μ the magnetic permeability, ν the kinematic viscosity and ρ the density of the fluid.

The velocity field $\mathbf{q}$ and the magnetic field $\mathbf{H}$ are given by

$$\mathbf{q}=\operatorname{curl}\psi(x,y)\,\mathbf{k}, \quad \mathbf{H}=\operatorname{curl}A(x,y)\,\mathbf{k}, \tag{1.4}$$

where $\mathbf{k}$ is a unit vector in the z-direction and

$$\psi=(U_0\nu x)^{\frac{1}{2}}f(\eta), \quad A=H_0(\nu x/U_0)^{\frac{1}{2}}g(\eta). \tag{1.5}$$

Thus f is associated with the velocity field and g with the magnetic field.

In their paper Greenspan & Carrier gave various forms of approximate solutions and also found some numerical solutions when $\epsilon=10$, 1, 0·05 and 0·005. Glauert (1961) has also found series solutions which are reliable for $\epsilon>10$ and $\epsilon<0{\cdot}001$; however, he assumed $1-\beta$ was not small and stated that these cannot be relied on near $\beta=1$. Reuter & Stewartson (1961) have shown that for $\beta>1$

there are no solutions of this boundary-value problem such that $f''(0) \geqslant 0$. For the present situation, therefore, β lies between 0 and 1.

Greenspan & Carrier discussed the behaviour of the solutions near $\beta = 1$ for $\epsilon = \infty$ and $\epsilon = 1$. An attempt by Stewartson to extend their work to cover all values of ϵ was successful for $\epsilon > 1$ but led to an apparent contradiction when $0 < \epsilon < 1$.† The present work was undertaken to throw some light on the nature of the solutions of the equations near $\beta = 1$ and to resolve the contradiction he found. The present paper is primarily concerned with the behaviour of solutions for $0{\cdot}75 < \beta < 1$ and certain values of ϵ between 0·01 and 10.

2. Method of solution

The equations were taken in the form

$$F''' + FF'' - GG'' = 0, \tag{2.1}$$

$$G'' + \epsilon(FG' - F'G) = 0, \tag{2.2}$$

and single-point boundary conditions used

$$F(0) = F'(0) = G(0) = 0, \quad G'(0) = q, \quad F''(0) = p, \tag{2.3}$$

the values of p and q being varied as required. Equations (2.1) and (2.2) were then written in the form of six first-order differential equations and integrated numerically on a computer using a Runge–Kutta method developed by Merson. This automatically adjusted the step length to allow for a predetermined truncation error. When F' and G' had reached their asymptotic values, say

$$F'(\infty) = 2A, \quad G'(\infty) = 2B,$$

the functions F and G were transformed by the relations

$$F(\eta) = A^{\frac{1}{2}}f(A^{\frac{1}{2}}\eta), \tag{2.4}$$

$$G(\eta) = A^{-\frac{1}{2}}Bg(A^{\frac{1}{2}}\eta). \tag{2.5}$$

These transform (2.1) and (2.2) into (1.1) and (1.2), satisfying (1.3), if $\beta = B^2/A^2$. Initial values of g' and f'' were then found: $g'(0) = q/B$, $f''(0) = p/A^{\frac{3}{2}}$. Thus a range of values of β was found with corresponding $g'(0)$ and $f''(0)$.

3. Duality of solution

Solutions were found with the parameter ϵ as 10, 0·5, 0·1 and 0·01. For $\epsilon = 10$ the plot of $f''(0)$ against β tended to the point (0, 1) as might be inferred from figure 2 of Greenspan & Carrier's paper and from Stewartson's work. However, for the other three values of ϵ, all less than unity, the plot of $f''(0)$ (and $g'(0)$) turned sharply round, never actually reaching $\beta = 1$, see figure 1. The values of β in each case attained an upper bound slightly less than unity; values are given in table 1. Thus for a range of values of β there were found to be two pairs of values of $f''(0)$ and $g'(0)$ satisfying (1.1), (1.2) and (1.3) for $\epsilon < 1$.

To examine the accuracy of the solutions various checks were carried out. It was mentioned in the previous section that the computational program

† The difference between the properties of f, g when $\epsilon < 1$ and when $\epsilon > 1$ is discussed in a forthcoming paper (Stewartson & Wilson 1964) which also includes an analytic explanation of the phenomenon described here.

used automatically adjusted the step length to allow for a predetermined truncation error in any of the variables calculated. This parameter was taken as 10^{-7}; in certain arbitrary cases where it was varied to 10^{-8} the proportional values of the functions computed were found to differ at the most by 10^{-5}.

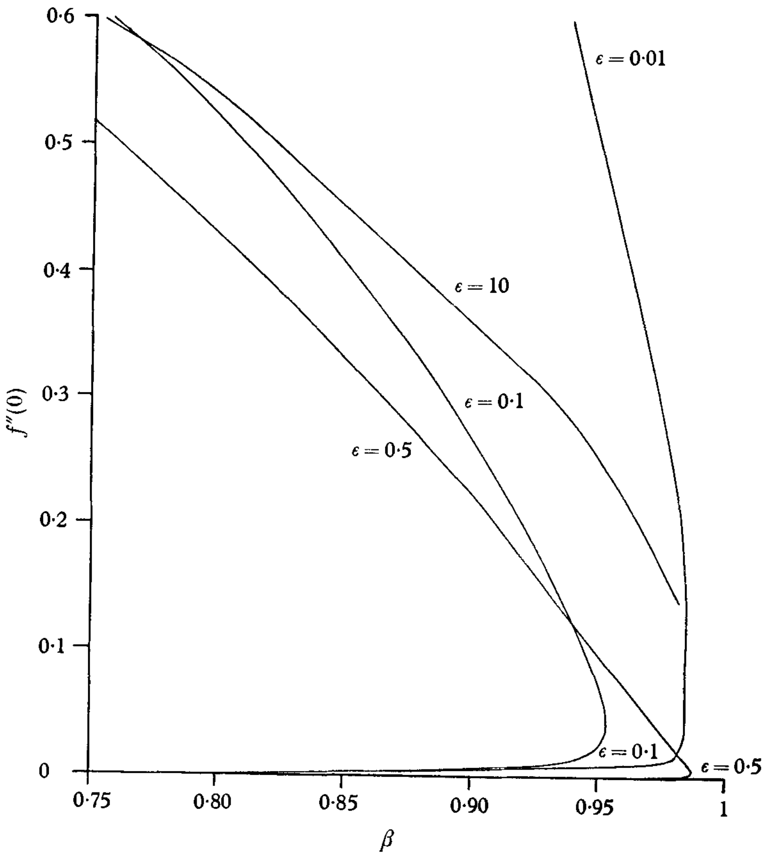

FIGURE 1. 'Skin friction', $f''(0)$, *vs* β for values of ϵ shown.

ϵ	Max. β	β at onset of inflections in velocity profile
0·01	0·984	0·94
0·1	0·953	0·84
0·5	0·988	0·88
10	—	0·92

TABLE 1

The computation was continued until $F''(\eta) < 10^{-5}$, when β was calculated; in cases of interest the computation was carried out again using equations (1.1) and (1.2) instead and the values just found for $\beta, f''(0)$ and $g'(0)$. In each case $f'(\infty)$ and $g'(\infty)$ came very close to 2; e.g. for $\epsilon = 0{\cdot}1$ the error on the upper part of the curve was between 0·005 and 0·1 % at maximum β and on the lower curve between 0·1 and 0·5 %. The lower accuracy here was attributable to the greater length of the calculation in both cases (due to the correspondingly wider boundary layer, see later) and the considerably smaller initial values of $f''(0)$ and $g'(0)$. Identical solutions were also obtained from different pairs of starting values $F''(0)$ and $G'(0)$.

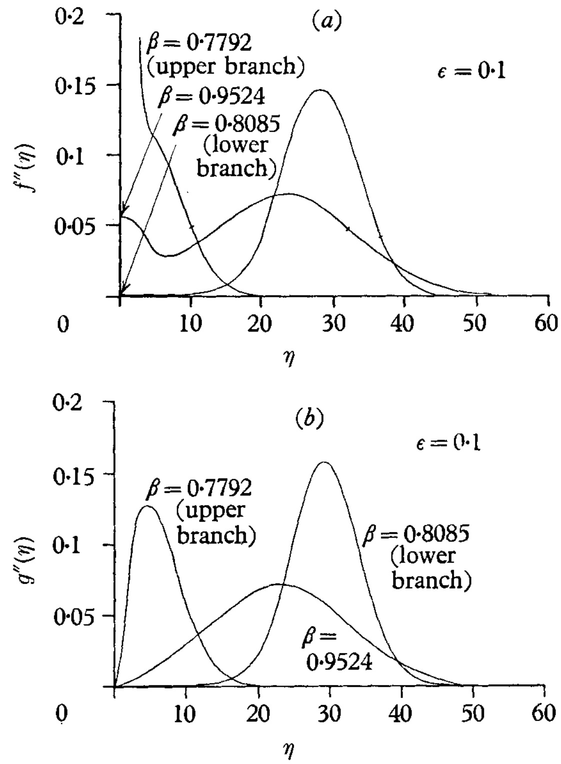

FIGURE 2. (a) $f''(\eta)$ vs η for three values of β when $\epsilon = 0·1$. To the right of points marked the curve follows closely the asymptotic behaviour. (b) $g''(\eta)$ vs η for 3 values of β when $\epsilon = 0·1$.

β	c	b	a	Valid for η greater than
0·7792	0·0209	0·120	3·5	10
0·9524	0·0045	0·084	20·8	32
0·8085	0·0185	0·163	28·0	37

TABLE 2

From (1.2) and (1.3) it can be seen (Glauert 1961) that, for any value of ϵ, when η is large $f \sim g \sim 2(\eta - a)$, where a is some constant. From (1.1) and (1.2) it can be shown that

$$f'' \sim b \exp\{-c(\eta-a)^2\}, \quad g'' \sim d \exp(-c(\eta-a)^2\}, \tag{3.1}$$

where b and d are constants and c is the algebraically smaller root of the equation

$$c^2 - (1+\epsilon)c + (1-\beta)\epsilon = 0. \tag{3.2}$$

Values of c were calculated for three of the points on the curve in figure 1 for $\epsilon = 0{\cdot}1$; one on the upper and one on the lower branch near $\beta = 0{\cdot}8$ and for the maximum value of β found. The computed behaviour of f'' for these three solutions is shown in figure 2. Values of a and b found for these appear in table 2. The deviation of the computed values from the asymptotic values was found to be less than 10^{-3} for η greater than the values given and to decrease with increasing η.

4. Discussion of results

Parallel to the plate the component of velocity is $u = \frac{1}{2}U_0 f'(\eta)$, and the component of the magnetic field $H_x = \frac{1}{2}H_0 g'(\eta)$. The velocity profiles for the three cases mentioned above, when $\epsilon = 0{\cdot}1$, are shown in figure 3. Two of

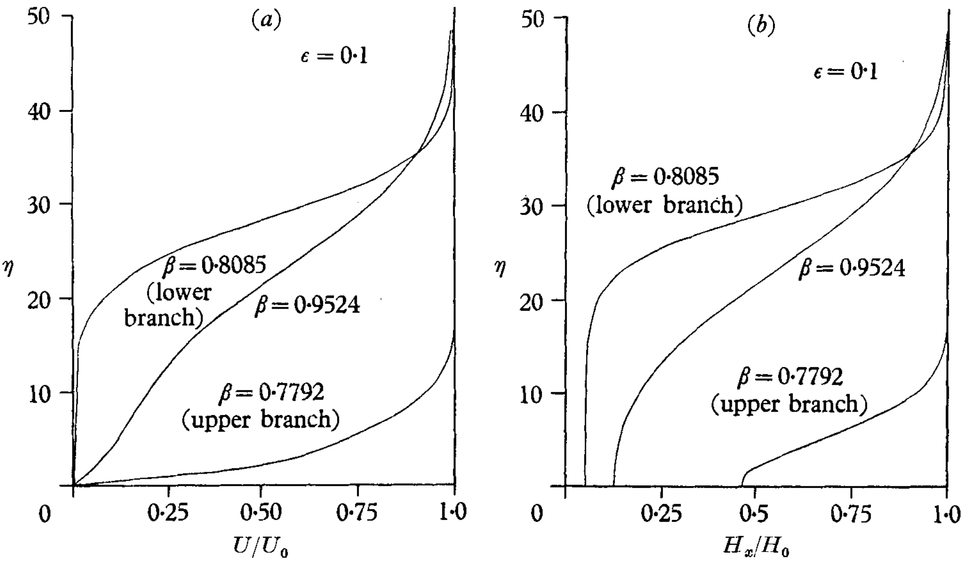

FIGURE 3. (a) Velocity profiles for three values of β when $\epsilon = 0{\cdot}1$. (b) Profiles of the magnetic field component H_x for three values of β when $\epsilon = 0{\cdot}1$.

these are seen to contain inflexions; were it not for the magnetic field this would imply instability in the boundary layer from Rayleigh's principle (see Schlichting 1955). Inflexions in the velocity profile occur where $f''' = 0$, and are the points where the tangents to the curves in figure 2 are parallel to the axis.

For each value of ϵ considered, inflexions in the velocity profile set in on the upper curve towards $\beta = 1$. Details are given in table 1.

The solutions on the lower curve, except near the turn, are considerably smaller than those on the upper curve for the same values of β. The drag on the plate is given by

$$\tau_w = \tfrac{1}{4}\rho(U_0^3\nu/x)^{\frac{1}{2}} f''(0), \tag{4.1}$$

and the current in the boundary layer by

$$H_t - H_0 = \tfrac{1}{2}H_0\{g'(0) - 2\}. \tag{4.2}$$

The smaller solutions therefore correspond to a considerably smaller drag and a larger current in the boundary layer.

For $\epsilon = 0{\cdot}1$ the width of the boundary layer increases from $\eta = 10$ for $\beta = 0{\cdot}425$ to $\eta = 20$ where inflexions first occur at $\beta = 0{\cdot}84$ to $\eta = 50$ at maximum $\beta = 0{\cdot}95$. It decreases slightly as β decreases on the lower branch of the curve in figure 1. The lower solutions have a boundary layer in general between two and five times thicker than the corresponding higher solutions. In each case the magnetic boundary layer is the same width as the fluid one.

For decreasing ϵ the graph of $f''(0)$ against β appears increasingly to approach the bounding lines $f''(0) = 1{\cdot}328$, $\beta = 1$, $f''(0) = 0$.

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